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SMOOTH CONVEX BODIES WITH PROPORTIONAL PROJECTION FUNCTIONS

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ABSTRACT

For a convex body $K \subset \mathbb{R}^n$ and $i \in \{1, \ldots, n-1\}$, the function assigning to any i-dimensional subspace L of $\mathbb{R}^n$, the i-dimensional volume of the orthogonal projection of K to L, is called the i-th projection function of K. Let $K, K_0 \subset \mathbb{R}^n$ be smooth convex bodies with boundaries of class C^2 and positive Gauss–Kronecker curvature and assume K_0 is centrally symmetric. Excluding two exceptional cases, $(i,j) = (1, n-1)$ and $(i,j) = (n-2, n-1)$, we prove that K and K_0 are homothetic if their i-th and j-th projection functions are proportional. When K_0 is a Euclidean ball this shows that a convex body with C^2 boundary and positive Gauss–Kronecker with constant i-th and j-th projection functions is a Euclidean ball.

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1. Introduction and main results

A **convex body** in $\mathbb{R}^n$ is a compact convex set with nonempty interior. If K is a convex body and L a linear subspace of $\mathbb{R}^n$, then $K|L$ is the orthogonal projection of K onto L. Let $\mathbb{G}(n,i)$ be the Grassmannian of all i-dimensional linear subspaces of $\mathbb{R}^n$. A central question in the geometric tomography of convex sets is to understand to what extent information about the projections $K|L$ with $L \in \mathbb{G}(n,i)$ determines a convex body. Possibly the most natural, but rather weak, information about $K|L$ is its i-dimensional volume $V_i(K|L)$. The function $L \mapsto V_i(K|L)$ on $\mathbb{G}(n,i)$ is the **i-th projection function** (or the **i-th brightness function**) of K. When $i = 1$ this is the **width function** and when $i = n-1$ the **brightness function**. If this function is constant, then the convex body K is said to have **constant i-brightness**. For $n \geq 2$ and any $i \in \{1, \ldots, n-1\}$, by classical results about the existence of sets with constant width and results of Blaschke [1, pp. 151–154] and Firey [6] there are nonspherical convex bodies of constant i-brightness (cf. [7, Thm 3.3.14, p. 111; Rmk 3.3.16, p. 114]). Corresponding examples of smooth convex bodies with everywhere positive Gauss–Kronecker curvature can be obtained by known approximation arguments (see [21, §3.3] and [12]). Thus it is not possible to determine if a convex body is a ball from just one projection function. For other results about determining convex bodies from a single projection function see Chapter 3 of Gardner's book [7] and the survey paper [10] of Goodey, Schneider, and Weil.

Therefore, as pointed out by Goodey, Schneider, and Weil in [10] and [11], it is natural to ask whether a convex body with two constant projection functions must be a ball. This question leads to the more general investigation of pairs of convex bodies, one of which is centrally symmetric, that have two of their projection functions proportional. Examples in the smooth and the polytopal setting, due to Campi [3], Gardner and Volčič [8], and to Goodey, Schneider, and Weil [11], show that the assumption of central symmetry on one of the bodies cannot be dropped. A convex body is said to be of class C^2_+ if its boundary, ∂K, is of class C^2 and has everywhere positive Gauss–Kronecker curvature. It is well-known that a convex body of class C^2_+ has a C^2 support function, but the converse need not be true. A classical result [20] of S. Nakajima (who also published under the name A. Matsumura) from 1926 states that a *three-dimensional* convex body of class C^2_+ with constant width and constant brightness is a Euclidean ball. This answers the previous question for smooth convex bodies in $\mathbb{R}^3$. Our main result generalizes Nakajima's theorem to the case of pairs of convex bodies with proportional projection functions, slightly relaxes the smoothness assumption, and, more importantly, provides an extension to higher dimensions.

THEOREM 1.1: *Let $K, K_0 \subset \mathbb{R}^n$ be convex bodies with K_0 of class C^2_+ and centrally symmetric and with K having C^2 support function. Let $1 \leq i < j \leq n-1$ be integers such that $i \notin \{1, n-2\}$ if $j = n-1$. Assume there are real positive constants $\alpha, \beta > 0$ such that*

$$V_i(K|L) = \alpha V_i(K_0|L) \quad \text{and} \quad V_j(K|U) = \beta V_j(K_0|U),$$

for all $L \in \mathbb{G}(n,i)$ and $U \in \mathbb{G}(n,j)$. Then K and K_0 are homothetic.

Other than Nakajima's result the only previously known case is $i = 1$ and $j = 2$ proven by Chakerian [4] in 1967. Letting K_0 be a Euclidean ball in the theorem, we get the following important special case.

COROLLARY 1.2: *Let $K \subset \mathbb{R}^n$ be a convex body with C^2 support function. Assume that K has constant i-brightness and constant j-brightness, where $1 \leq i < j \leq n-1$ and $i \notin \{1, n-2\}$ if $j = n-1$. Then K is a Euclidean ball.*

If ∂K is of class C^2 and K has constant width, then the Gauss–Kronecker curvature of K is everywhere positive. Thus we can conclude that K is of class C^2_+, which yields the following corollary.

COROLLARY 1.3: *Let $K \subset \mathbb{R}^n$ be a convex body of class C^2 with constant width and constant k-brightness for some $k \in \{2, \ldots, n-2\}$. Then K is a Euclidean ball.*

Corollary 1.3 does not cover the case that K has constant width and brightness, which we consider the most interesting open problem related to the subject of this paper. Under the strong additional assumption that K and K_0 are smooth convex bodies of revolution with a common axis, we can also settle the two cases not covered by Theorem 1.1.

PROPOSITION 1.4: *Let $K, K_0 \subset \mathbb{R}^n$ be convex bodies that have a common axis of revolution such that K has C^2 support function and K_0 is centrally symmetric and of class C^2_+. Assume that K and K_0 have proportional brightness and proportional i-th brightness function for an $i \in \{1, n-2\}$. Then K is homothetic to K_0. In particular, if K_0 is a Euclidean ball, then K is also a Euclidean ball.*

From the point of view of convexity theory the restriction to convex bodies of class C^2_+ or with C^2 support functions is not natural and it would be of great interest to extend Theorem 1.1 and Corollaries 1.2 and 1.3 to general convex bodies. In the case of Corollary 1.3 when $n \geq 3$, $i = 1$ and $j = 2$ this was done in [15]. However, from the point of view of differential geometry, the class C^2_+ is quite natural and the convex bodies of constant i-brightness in C^2_+ have some interesting differential geometric properties. If ∂K is a C^2 hypersurface, then (as usual) $x \in \partial K$ is called an **umbilic point** of K if all of the principal curvatures of ∂K at x are equal. In the C^2_+ case, this is equivalent to the condition that all of the principal radii of curvature of K at the outer unit normal vector of K at x are equal. The following is a special case of Proposition 5.2 below.

Proposition 1.5: *Let K be a convex body of class C^2_+ in $\mathbb{R}^n$ with $n \geq 5$, and let $2 \leq k \leq n-3$. Assume that K has constant k-brightness. Then ∂K has a pair of umbilic points x_1 and x_2 such that the tangent planes of ∂K at x_1 and x_2 are parallel and all of the principal curvatures of ∂K at x_1 and x_2 are equal.*

This is surprising because when $n \geq 4$ the set of convex bodies of class C^2_+ with no umbilic points is a dense open set in C^2_+ with the C^2 topology.

Finally, we comment on the relation of our results to those in the paper [14] of Haab. All our main results are stated by Haab, but his proofs are either incomplete or have errors (see the review in Zentralblatt). In particular, the proof of his main result, stating that a convex body of class C^2_+ with constant width and constant $(n-1)$-brightness is a ball, is wrong (the proof is based on [14, Lemma 5.3] which is false even in the case of $n = 1$) and this case is still open. We have included remarks at the appropriate places relating our results and proofs to those in [14]. Despite the errors in [14], the paper still has some important insights. In particular, while Haab's proof of his Theorem 4.1 (our Proposition 3.5) is incomplete, see Remark 3.2 below, the statement is correct and is the basis for the proofs of most of our results. Also it was Haab who realized that having constant brightness implies the existence of umbilic points. While his proof is incomplete and the details of the proof here differ a good deal from those of his proposed argument, the global structure of the proof here is still indebted to his paper.

2. Preliminaries

We will work in the Euclidean space $\mathbb{R}^n$ with the usual inner product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. The support function of a convex body K in $\mathbb{R}^n$ is the function $h_K \colon \mathbb{R}^n \to \mathbb{R}$ given by $h_K(x) = \max_{y \in K} \langle x, y \rangle$. The function h_K is homogeneous of degree one. A convex body is uniquely determined by its support function. Subsequently, we summarize some facts from [21] which are needed. An important fact for us, first noted by Wintner [22, Appendix], is that if K is of class C^2_+, then its support function h_K is of class C^2 on $\mathbb{R}^n \setminus \{0\}$ and the principal radii of curvature (see below for a definition) of K are everywhere positive (cf. [21, p. 106]). Conversely, if the support function of K is of class C^2 on $\mathbb{R}^n \setminus \{0\}$ and the principal radii of curvature of K are everywhere positive, then K is of class C^2_+ (cf. [21, p. 111]). In this paper, we say that a support function is of class C^2 if it is of class C^2 on $\mathbb{R}^n \setminus \{0\}$. Let L be a linear subspace of $\mathbb{R}^n$. Then the support function of the projection $K|L$ is the restriction $h_{K|L} = h_K|_L$. In particular, if h_K is of class C^2, then $h_{K|L}$ is of class C^2 in L. As an easy consequence we obtain that if K is of class C^2_+, then $K|L$ is of class C^2_+ in L.

All of our proofs work for convex bodies $K \subset \mathbb{R}^n$ that have a C^2 support function. That this leads to a genuine extension of the C^2_+ setting can be seen from the following example. Let K be of class C^2_+ and let r_0 be the minimum of all of the principal radii of curvature of ∂K. Then by Blaschke's rolling theorem (cf. [21, Thm 3.2.9, p. 149]) there is a convex set K_1 and a ball B_{r_0} of radius r_0 such that K is the Minkowski sum $K = K_1 + B_{r_0}$ and no ball of radius greater than r_0 is a Minkowski summand of K. Thus no ball is a summand of K_1, for if $K_1 = K_2 + B_r$, $r > 0$, then $K = K_1 + B_{r_0} = K_2 + B_{r+r_0}$, contradicting the maximality of r_0. As every convex body with C^2 boundary has a ball as a summand, it follows that K_1 does not have a C^2 boundary. But the support function of K_1 is $h_{K_1} = h_K - r_0|\cdot|$ and therefore h_{K_1} is C^2. When K_1 has nonempty interior, for example when K is an ellipsoid with all axes of different lengths, then K_1 is an example of a convex set with C^2 support function, but with ∂K_1 not of class C^2.

If the support function $h = h_K$ of a convex body $K \subset \mathbb{R}^n$ is of class C^2, then let $\operatorname{grad} h_K$ be the usual gradient of h_K. This is a C^1 vector field on $\mathbb{R}^n \setminus \{0\}$ (which is homogeneous of degree zero). Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$. Then for $u \in \mathbb{S}^{n-1}$ the unique point on ∂K with outward unit normal u is $\operatorname{grad} h_K(u)$ (cf. [21, (2.5.8), p. 107]). In the case where K is of class C^2_+, the map $\mathbb{S}^{n-1} \to \partial K$, $u \mapsto \operatorname{grad} h_K(u)$, is the inverse of the **spherical image map** (Gauss map) of K. For this reason, this map is called the **reverse spherical image map** (cf. [21, p. 107]) of K whenever h_K is of class C^2. Let $d^2 h_K$ be the usual Hessian of h_K viewed as a field of selfadjoint linear maps on $\mathbb{R}^n \setminus \{0\}$. That is, for $u \in \mathbb{R}^n \setminus \{0\}$ and $x \in \mathbb{R}^n$, $d^2 h_K(u)x$ is the directional derivative of $\operatorname{grad} h_K$ at u in the direction x. As h_K is homogeneous of degree one, for any $u \in \mathbb{S}^{n-1}$ it follows that $d^2 h_K(u)u = 0$. Since $d^2 h_K(u)$ is selfadjoint, this implies that the orthogonal complement $u^\perp$ of u is invariant under $d^2h_K(u)$. As $u^\perp = T_u\mathbb{S}^{n-1}$ we can then define a field of selfadjoint linear maps $L(h_K)$ on the tangent spaces to $\mathbb{S}^{n-1}$ by

$$L(h_K)(u) := d^2h_K(u)\big|_{u^\perp}.$$

Clearly, $L(h_K)(u)$ can (and occasionally will) be identified with a symmetric bilinear form on $u^\perp$, via the scalar product induced on $u^\perp$ from $\mathbb{R}^n$. For given $u \in \mathbb{S}^{n-1}$, $L(h_K)(u)$ is the **reverse Weingarten map** of K at u. The eigenvalues of $L(h_K)(u)$ are the **principal radii of curvature** of K at u (cf. [21, p. 108]). Due to the convexity of the support function, these are nonnegative real numbers (the corresponding bilinear form is positive semidefinite). Recall that if K is of class C^2_+, the derivative of the Gauss map of K at $x \in \partial K$ is the **Weingarten map** of K at x. This is a selfadjoint linear map of the tangent space of ∂K at x whose eigenvalues are called the **principal curvatures** of K at x. In the C^2_+ case, $L(h_K)(u)$ is the inverse of the Weingarten map of K at $x = \operatorname{grad} h_K(u)$, for any $u \in \mathbb{S}^{n-1}$, and both maps are positive definite.

In the following, the notion of the (surface) area measure of a convex body will be useful. In the case of general convex bodies the definition is a bit involved, see [21, pp. 200–203] or [7, pp. 351–353], but we will only need the case of bodies with support functions of class C^2 where an easier definition is possible. Let $K \subset \mathbb{R}^n$ be a convex body with support function h_K of class C^2. Then the (top order) **surface area measure** $S_{n-1}(K,\cdot)$ of K is defined on Borel subsets ω of $\mathbb{S}^{n-1}$ by

$$S_{n-1}(K,\omega) := \int_\omega \det(L(h_K)(u))du, \tag{2.1}$$

where du denotes integration with respect to spherical Lebesgue measure. (See, for instance, [21, (4.2.20), p. 206; Chap. 5] or [7, (A.7), p. 353].)

We need also a generalization of the operator $L(h_K)$. Let $K_0 \subset \mathbb{R}^n$ be a convex body of class C^2_+, and let h_0 be the support function of K_0. As K_0 is of class C^2_+, the linear map $L(h_0)(u)$ is positive definite for all $u \in \mathbb{S}^{n-1}$. Therefore $L(h_0)(u)$ will have a unique positive definite square root which we denote by $L(h_0)^{1/2}(u)$. Then for any convex body $K \subset \mathbb{R}^n$ with support function h_K of class C^2, we define

$$L_{h_0}(h_K)(u) := L(h_0)^{-1/2}(u)L(h_K)(u)L(h_0)^{-1/2}(u) \tag{2.2}$$

where $L(h_0)^{-1/2}(u)$ is the inverse of $L(h_0)^{1/2}(u)$. It is easily checked that if K is of class C^2_+, then $L_{h_0}(h_K)(u)$ is positive definite for all u. Furthermore, we always have

$$\det(L_{h_0}(h_K)(u)) = \frac{\det(L(h_K)(u))}{\det(L(h_0)(u))}.$$

The linear map $L_{h_0}(h_K)(u)$ has the interpretation as the inverse Weingarten map in the relative geometry defined by K_0. This interpretation will not be used in the present paper, but it did motivate some of the calculations.

3. Projections and support functions

3.1. Some multilinear algebra. The geometric condition of proportional projection functions can be translated into a condition involving reverse Weingarten maps. In order to fully exploit this information, the following lemmas will be used. In fact, these lemmas fill a gap in [14, §4]. For basic results concerning the Grassmann algebra and alternating maps, which are used subsequently, we refer to [17], [18].

Lemma 3.1: *Let $G, H, L\colon \mathbb{R}^n \to \mathbb{R}^n$ be positive semidefinite linear maps. Let $k \in \{1, \dots, n\}$, and assume that*

$$\langle(\wedge^k G + \wedge^k H)\xi, \xi\rangle = \langle(\wedge^k L)\xi, \xi\rangle \tag{3.1}$$

for all decomposable $\xi \in \bigwedge^k \mathbb{R}^n$. Then

$$\wedge^k G + \wedge^k H = \wedge^k L. \tag{3.2}$$

Proof: It is sufficient to consider the cases $k \in \{2, \dots, n-1\}$. For $\xi, \zeta \in \bigwedge^k \mathbb{R}^n$, we define

$$\omega_L(\xi, \zeta) := \langle(\wedge^k L)\xi, \zeta\rangle.$$

Then, for any $u_1, \dots, u_{k+1}, v_1, \dots, v_{k-1} \in \mathbb{R}^n$, the identity

$$\sum_{j=1}^{k+1}(-1)^j \omega_L(u_1 \wedge \cdots \wedge \check{u}_j \wedge \cdots \wedge u_{k+1}; u_j \wedge v_1 \wedge \cdots \wedge v_{k-1}) = 0 \tag{3.3}$$

is satisfied, where $\check{u}_j$ means that u_j is omitted. Thus, in the terminology of [16], ω_L satisfies the first Bianchi identity. Once (3.3) has been verified, the proof of Lemma 3.1 can be completed as follows. Define ω_G and ω_H by replacing L in the definition of ω_L by G and H, respectively. Then $\omega_{G,H} := \omega_G + \omega_H$ also satisfies the first Bianchi identity. By assumption,

$$\omega_{G,H}(\xi, \xi) = \omega_L(\xi, \xi)$$

for all decomposable $\xi \in \bigwedge^k \mathbb{R}^n$. Proposition 2.1 in [16] now implies that

$$\omega_{G,H}(\xi, \zeta) = \omega_L(\xi, \zeta)$$

for all decomposable $\xi, \zeta \in \bigwedge^k \mathbb{R}^n$, which yields the assertion of the lemma.

For the proof of (3.3) we proceed as follows. Since L is positive semidefinite, there is a positive semidefinite linear map φ: $\mathbb{R}^n \to \mathbb{R}^n$ such that $L = \varphi \circ \varphi$. Hence

$$\omega_L(u_1 \wedge \cdots \wedge u_k; v_1 \wedge \cdots \wedge v_k) = \langle \varphi u_1 \wedge \cdots \wedge \varphi u_k, \varphi v_1 \wedge \cdots \wedge \varphi v_k \rangle$$

for all $u_1, \ldots, v_k \in \mathbb{R}^n$. For $a_1, \ldots, a_{k+1}, b_1, \ldots, b_{k-1} \in \mathbb{R}^n$ we define

$$\begin{aligned}\Phi(a_1, &\ldots, a_{k+1}; b_1, \ldots, b_{k-1}) \\ &:= \sum_{j=1}^{k+1} (-1)^j \langle a_1 \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_{k+1}; a_j \wedge b_1 \wedge \cdots \wedge b_{k-1} \rangle.\end{aligned}$$

We will show that $\Phi = 0$. Then, substituting $a_i = \varphi(u_i)$ and $b_j = \varphi(v_j)$, we obtain the required assertion (3.3).

For the proof of $\Phi = 0$, it is sufficient to show that Φ vanishes on the vectors of an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, since Φ is a multilinear map. So let $a_1, \ldots, a_{k+1} \in \{e_1, \ldots, e_n\}$, whereas $b_1, \ldots, b_{k-1}$ are arbitrary.

If $a_1, \ldots, a_{k+1}$ are distinct, then the summands of Φ vanish, since $\langle a_i, a_j \rangle = 0$ for $i \neq j$. Here we use that

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle_{i,j=1}^k)$$

for $u_1, \ldots, u_k, v_1, \ldots, v_k \in \mathbb{R}^n$.

Otherwise, $a_i = a_j$ for some $i \neq j$. In this case, we argue as follows. Without loss of generality we assume that $i < j$. Then, repeatedly using that $a_i = a_j$, we get

$$\begin{aligned}\Phi(a_1, &\ldots, a_{k+1}; b_1, \ldots, b_{k-1}) \\ =&(-1)^i \langle a_1 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_j \wedge \cdots \wedge a_{k+1}; a_i \wedge b_1 \wedge \cdots \wedge b_{k-1} \rangle \\ &+ (-1)^j \langle a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_{k+1}; a_j \wedge b_1 \wedge \cdots \wedge b_{k-1} \rangle \\ =&(-1)^i (-1)^{j-i-1} \langle a_1 \wedge \cdots \wedge a_j \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_{k+1}; a_i \wedge b_1 \wedge \cdots \wedge b_{k-1} \rangle \\ &+ (-1)^j \langle a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_{k+1}; a_j \wedge b_1 \wedge \cdots \wedge b_{k-1} \rangle \\ =&0,\end{aligned}$$

which completes the proof. ∎

Remark 3.2: In the proof of Theorem 4.1 in [14], Haab uses a special case of Lemma 3.1, but his proof is incomplete. To describe the situation more carefully, let $T\colon \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n$ denote a symmetric linear map satisfying $\langle T\xi, \xi\rangle = 1$ for all decomposable unit vectors $\xi \in \bigwedge^k \mathbb{R}^n$. From this hypothesis Haab apparently concludes that T is the identity map (cf. [14, p. 126, l. 15-20]). While Lemma 3.1 implies that a corresponding fact is indeed true for maps T of a special form, a counterexample for the general assertion is provided in [18, pp. 124-5]. For a different counterexample, let k be even and let Q be the symmetric bilinear form defined on $\bigwedge^k(\mathbb{R}^{2k})$ by $Q(w,w) = w \wedge w$. This is a symmetric bilinear form as k is even and $w \wedge w \in \bigwedge^{2k} \mathbb{R}^{2k}$ so that $\bigwedge^{2k} \mathbb{R}^{2k}$ is one dimensional and thus can be identified with the real numbers. In this example, $Q(\xi,\xi) = 0$ for all decomposable k-vectors ξ, but Q is not the zero bilinear form.

Remark 3.3: Haab states a (simpler) version of the next lemma, [14, Cor 4.2, p. 126], without proof.

Lemma 3.4: *Let $G, H\colon \mathbb{R}^n \to \mathbb{R}^n$ be selfadjoint linear maps and assume that*

$$\wedge^k G + \wedge^k H = \beta \wedge^k \mathrm{id}$$

for some constant $\beta \in \mathbb{R}$ with $\beta \neq 0$ and some $k \in \{1, \dots n-1\}$. Then G and H have a common orthonormal basis of eigenvectors. If $k \geq 2$, then either G or H is an isomorphism.

Proof: If $k = 1$, this is elementary so we assume that $2 \leq k \leq n-1$. We first show that at least one of G or H is nonsingular. Assume that this is not the case. Then both the kernels $\ker G$ and $\ker H$ have positive dimension. Choose k linearly independent vectors $v_1, \dots, v_k$ as follows: If $\ker G \cap \ker H \neq \{0\}$, then let $0 \neq v_1 \in \ker G \cap \ker H$ and choose vectors $v_2, \dots, v_k$ so that $v_1, v_2, \dots, v_k$ are linearly independent. If $\ker G \cap \ker H = \{0\}$, then there are nonzero $v_1 \in \ker G$ and $v_2 \in \ker H$. Then $\ker G \cap \ker H = \{0\}$ implies that v_1 and v_2 are linearly independent. So in this case choose $v_3, \dots, v_k$ so that $v_1, \dots, v_k$ are linearly independent. In either case

$$\begin{aligned}
(\wedge^k G + \wedge^k H)&v_1 \wedge v_2 \wedge \cdots \wedge v_k \\
&= Gv_1 \wedge Gv_2 \wedge \cdots \wedge Gv_k + Hv_1 \wedge Hv_2 \wedge \cdots \wedge Hv_k \\
&= 0
\end{aligned}$$

which contradicts that $\wedge^k G + \wedge^k H = \beta \wedge^k \mathrm{id}$ and $\beta \neq 0$.

Without loss of generality we assume that H is nonsingular. Since G is selfadjoint, there exists an orthonormal basis $e_1, \dots, e_n$ of eigenvectors of G

with corresponding eigenvalues $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. For a decomposable vector $\xi = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbb{R}^n \setminus \{0\}$, we define

$$[\xi] := \operatorname{span}\{v \in \mathbb{R}^n : v \wedge \xi = 0\} = \operatorname{span}\{v_1, \ldots, v_k\} \in \mathbb{G}(n,k).$$

Then, for any $1 \le i_1 < \ldots < i_k \le n$, we get

$$\begin{aligned} H(\operatorname{span}\{e_{i_1}, \ldots, e_{i_k}\}) &= \operatorname{span}\{H(e_{i_1}), \ldots, H(e_{i_k})\} = [H(e_{i_1}) \wedge \cdots \wedge H(e_{i_k})] \\ &= [(\wedge^k H) e_{i_1} \wedge \cdots \wedge e_{i_k}] \\ &= [(\beta \wedge^k \mathrm{id} - \wedge^k G) e_{i_1} \wedge \cdots \wedge e_{i_k}] \\ &= [(\beta - \alpha_{i_1} \cdots \alpha_{i_k}) e_{i_1} \wedge \cdots \wedge e_{i_k}] \\ &= \operatorname{span}\{e_{i_1}, \ldots, e_{i_k}\}, \end{aligned}$$

where we used that H is an isomorphism to obtain the second and the last equality. Since $k \le n-1$, we can conclude that

$$\begin{aligned} H(\operatorname{span}\{e_1\}) &= H\left(\bigcap_{j=2}^{k+1} \operatorname{span}\{e_1, \ldots, \check{e}_j, \ldots, e_{k+1}\} \right) \\ &= \bigcap_{j=2}^{k+1} H(\operatorname{span}\{e_1, \ldots, \check{e}_j, \ldots, e_{k+1}\}) \\ &= \bigcap_{j=2}^{k+1} \operatorname{span}\{e_1, \ldots, \check{e}_j, \ldots, e_{k+1}\} \\ &= \operatorname{span}\{e_1\}. \end{aligned}$$

By symmetry, we obtain that e_i is an eigenvector of H for $i = 1, \ldots, n$. ∎

3.2. One proportional projection function. Subsequently, if $K, K_0 \subset \mathbb{R}^n$ are convex bodies with support functions of class C^2, we put $h := h_K$ and $h_0 := h_{K_0}$ to simplify our notation. The following proposition is basic for the proofs of our main results.

Proposition 3.5: *Let $K, K_0 \subset \mathbb{R}^n$ be convex bodies having support functions of class C^2, let K_0 be centrally symmetric, and let $k \in \{1, \ldots, n-1\}$. Assume that $\beta > 0$ is a positive constant such that*

$$V_k(K|U) = \beta V_k(K_0|U) \tag{3.4}$$

for all $U \in \mathbb{G}(n,k)$. Then, for all $u \in \mathbb{S}^{n-1}$,

$$\wedge^k L(h)(u) + \wedge^k L(h)(-u) = 2\beta \wedge^k L(h_0)(u). \tag{3.5}$$

Proof: Let $u \in \mathbb{S}^{n-1}$ and a decomposable unit vector $\xi \in \bigwedge^k T_u\mathbb{S}^{n-1}$ be fixed. Then there exist orthonormal vectors $e_1, \dots, e_k \in u^\perp$ such that $\xi = e_1 \wedge \cdots \wedge e_k$. Put $E := \operatorname{span}\{e_1, \dots, e_k, u\} \in \mathbb{G}(n, k+1)$ and $E_0 := \operatorname{span}\{e_1, \dots, e_k\} \in \mathbb{G}(n, k)$. For any $v \in E \cap \mathbb{S}^{n-1}$,

$$V_k((K|E)|(v^\perp \cap E)) = \beta V_k((K_0|E)|(v^\perp \cap E)),$$

and therefore a special case of Theorem 2.1 in [9] (see also Theorem 3.3.2 in [7]) yields that

$$S_k^E(K|E, \cdot) + S_k^E((K|E)^*, \cdot) = 2\beta S_k^E(K_0|E, \cdot),$$

where $S_k^E(M, \cdot)$ denotes the (top order) surface area measure of a convex body M in E, and $(K|E)^*$ is the reflection of $K|E$ through the origin. Since $h_{K|E} = h_K\big|_E$ is of class C^2 in E, equation (2.1) applied in E implies that

$$\det\left(d^2 h_{K|E}(u)\big|_{E_0}\right) + \det\left(d^2 h_{K|E}(-u)\big|_{E_0}\right) = 2\beta \det\left(d^2 h_{K_0|E}(u)\big|_{E_0}\right). \tag{3.6}$$

Since $e_1, \dots, e_k, u$ is an orthonormal basis of E, we further deduce that

$$\begin{aligned}\det\left(d^2 h_{K|E}(u)\big|_{E_0}\right) &= \det(d^2 h_K(u)(e_i, e_j)_{i,j=1}^k) = \det(\langle L(h)(u)e_i, e_j\rangle_{i,j=1}^k) \\ &= \langle \wedge^k L(h)(u)\xi, \xi\rangle,\end{aligned}$$

and similarly for the other determinants. Substituting these expressions into (3.6) yields that

$$\langle(\wedge^k L(h)(u) + \wedge^k L(h)(-u))\xi, \xi\rangle = \langle 2\beta \wedge^k L(h_0)(u)\xi, \xi\rangle$$

for all decomposable (unit) vectors $\xi \in \bigwedge^k T_u\mathbb{S}^{n-1}$. Hence the required assertion follows from Lemma 3.1. ∎

It is useful to rewrite Proposition 3.5 in the notation of (2.2). The following corollary is implied by Proposition 3.5 and Lemma 3.4.

Corollary 3.6: *Let $K, K_0 \subset \mathbb{R}^n$ be convex bodies with K_0 being centrally symmetric and of class C^2_+ and K having C^2 support function. Let $k \in \{1, \dots, n-1\}$. Assume that $\beta > 0$ is a positive constant such that*

$$V_k(K|U) = \beta V_k(K_0|U)$$

for all $U \in \mathbb{G}(n, k)$. Then, for all $u \in \mathbb{S}^{n-1}$,

$$\wedge^k L_{h_0}(h)(u) + \wedge^k L_{h_0}(h)(-u) = 2\beta \wedge^k \mathrm{id}_{T_u\mathbb{S}^{n-1}}. \tag{3.7}$$

Moreover, for $k \in \{1, \dots, n-2\}$ the linear maps $L_{h_0}(h)(u)$ and $L_{h_0}(h)(-u)$ have a common orthonormal basis of eigenvectors.

4. The cases $1 \le i < j \le n-2$

4.1. Polynomial relations. In the sequel, it will be convenient to use the following notation. If $x_1, \ldots, x_n$ are nonnegative real numbers and $I \subset \{1, \ldots, n\}$, then we put

$$x_I := \prod_{\iota \in I} x_\iota .$$

If $I = \varnothing$, the empty product is interpreted as $x_\varnothing := 1$. The cardinality of the set I is denoted by $|I|$.

Lemma 4.1: *Let $a, b > 0$ and $2 \le k < m \le n-1$ with $a^m \ne b^k$. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be positive real numbers such that*

$$x_I + y_I = 2a \quad \text{and} \quad x_J + y_J = 2b$$

whenever $I, J \subset \{1, \ldots, n\}$, $|I| = k$ and $|J| = m$. Then there is a constant $c > 0$ such that $x_\iota / y_\iota = c$ for $\iota = 1, \ldots, n$.

Proof: It is easy to see that this can be reduced to the case where $m = n-1$. Thus we assume that $m = n-1$. By assumption,

$$x_\iota x_I + y_\iota y_I = 2a \quad \text{and} \quad x_\iota x_{I'} + y_\iota y_{I'} = 2a$$

whenever $\iota \in \{1, \ldots, n\}$, $I, I' \subset \{1, \ldots, n\} \setminus \{\iota\}$, $|I| = |I'| = k-1$. Subtracting these two equations, we get

$$x_\iota (x_I - x_{I'}) = y_\iota (y_{I'} - y_I). \tag{4.1}$$

By symmetry, it is sufficient to prove that $x_1/y_1 = x_2/y_2$. We distinguish several cases.

Case 1: There exist $I, I' \subset \{3, \ldots, n\}$, $|I| = |I'| = k-1$ with $x_I \ne x_{I'}$. Then (4.1) implies that

$$\frac{x_1}{y_1} = \frac{y_{I'} - y_I}{x_I - x_{I'}} = \frac{x_2}{y_2}.$$

Case 2: For all $I, I' \subset \{3, \ldots, n\}$ with $|I| = |I'| = k-1$, we have $x_I = x_{I'}$.

Since $1 \le k-1 \le n-3$, we obtain $x := x_3 = \cdots = x_n$. From (4.1) we get that also $y_I = y_{I'}$ for all $I, I' \subset \{3, \ldots, n\}$ with $|I| = |I'| = k-1$. Hence, $y := y_3 = \cdots = y_n$.

Case 2.1: $x_1 = x_2$. Since

$$x_1 x^{k-1} + y_1 y^{k-1} = 2a, \quad x_2 x^{k-1} + y_2 y^{k-1} = 2a$$

and $x_1 = x_2$, it follows that $y_1 = y_2$. In particular, we have $x_1/y_1 = x_2/y_2$.

Case 2.2: $x_1 \neq x_2$.
Case 2.2.1: x_1, x_2, x_3 are mutually distinct. Choose

$$I := \{2\} \cup \{5, 6, \ldots, k+2\}, \quad I' := \{4\} \cup \{5, 6, \ldots, k+2\}.$$

Note that here $k+2 \leq n$ and $\{5, 6, \ldots, k+2\}$ is the empty set for $k = 2$. Then $x_I \neq x_{I'}$ as $x_2 \neq x_4 = x_3$. Hence (4.1) yields that

$$\frac{x_1}{y_1} = \frac{y_{I'} - y_I}{x_I - x_{I'}} = \frac{x_3}{y_3}. \tag{4.2}$$

Next choose

$$I := \{1\} \cup \{5, 6, \ldots, k+2\}, \quad I' := \{4\} \cup \{5, 6, \ldots, k+2\}.$$

Then $x_I \neq x_{I'}$ as $x_1 \neq x_4 = x_3$, and hence (4.1) yields that

$$\frac{x_2}{y_2} = \frac{y_{I'} - y_I}{x_I - x_{I'}} = \frac{x_3}{y_3}. \tag{4.3}$$

From (4.2) and (4.3), we get $x_1/y_1 = x_2/y_2$.
Case 2.2.2: $x_1 \neq x_2 = x_3$ or $x_1 = x_3 \neq x_2$. By symmetry, it is sufficient to consider the first case. Since $k-1 \leq n-3$ and using

$$x_2 x^{k-1} + y_2 y^{k-1} = 2a \quad \text{and} \quad x_3 x^{k-1} + y_3 y^{k-1} = 2a,$$

we get $y_2 = y_3$. By the assumption of the proposition, the equations

$$x_2^k + y_2^k = 2a, \tag{4.4}$$

$$x_1 x_2^{k-1} + y_1 y_2^{k-1} = 2a, \tag{4.5}$$

$$x_2^{n-1} + y_2^{n-1} = 2b, \tag{4.6}$$

$$x_1 x_2^{n-2} + y_1 y_2^{n-2} = 2b. \tag{4.7}$$

are satisfied. From (4.4) and (4.5), we get

$$x_2^{k-1}(x_2 - x_1) + y_2^{k-1}(y_2 - y_1) = 0.$$

Moreover, (4.6) and (4.7) imply that

$$x_2^{n-2}(x_2 - x_1) + y_2^{n-2}(y_2 - y_1) = 0.$$

Since $x_1 \neq x_2$, we thus obtain

$$\frac{y_1 - y_2}{x_2 - x_1} = \frac{x_2^{k-1}}{y_2^{k-1}} = \frac{x_2^{n-2}}{y_2^{n-2}},$$

and therefore $y_2/x_2 = 1$. But now (4.4), (4.6) and $x_2 = y_2$ give $x_2^k = a$ and $x_2^{n-1} = b$, hence $a^{n-1} = b^k$, a contradiction. Thus Case 2.2.2 cannot occur. ∎

LEMMA 4.2: *Let $a, b > 0$ and $1 \le k < m \le n-1$ with $a^m \ne b^k$. Then there exists a finite set $\mathcal{F} = \mathcal{F}_{a,b,k,m}$, depending only on a, b, k, m, such that the following is true: if $x_1, \dots, x_n$ are nonnegative and $y_1, \dots, y_n$ are positive real numbers such that*

$$x_I + y_I = 2a \quad \text{and} \quad x_J + y_J = 2b$$

whenever $I, J \subset \{1, \dots, n\}$, $|I| = k$ and $|J| = m$, then $y_1, \dots, y_n \in \mathcal{F}$.

Remark 4.3: The condition $a^m \ne b^k$ is necessary in this lemma. For example, if $a = b = 1$, let $x_1 = x_2 = \cdots = x_{n-1} = y_1 = y_2 = \cdots = y_{n-1} = 1$, $x_n = t$ and $y_n = 1 - t$, where $t \in (0, 1)$. Then $x_I + y_I = 2$ for any nonempty subset I of $\{1, \dots, n\}$.

Proof: It is easy to see that it is sufficient to consider the case $m = n - 1$.

First, we consider the case $k = 1$. Moreover, we assume that $x_1, \dots, x_n$ are positive. Then, by assumption,

$$x_\iota + y_\iota = 2a \quad \text{and} \quad x_J + y_J = 2b \tag{4.8}$$

for $\iota = 1, \dots, n$ and $J \subset \{1, \dots, n\}$, $|J| = n - 1$. We put $X := x_{\{1,\dots,n\}}$ and $Y := y_{\{1,\dots,n\}}$. Then (4.8) implies

$$\frac{X}{x_\ell} + \frac{Y}{y_\ell} = 2b, \quad \ell = 1, \dots, n.$$

Using $y_\ell = 2a - x_\ell$, this results in

$$2bx_\ell^2 + (-X + Y - 4ab)x_\ell + 2aX = 0.$$

The quadratic equation

$$2bz^2 + (-X + Y - 4ab)z + 2aX = 0$$

has at most two real solutions z_1, z_2, hence $x_1, \dots, x_n \in \{z_1, z_2\}$.

CASE 1: $x_1 = \cdots = x_n =: x$. Then by (4.8) also $y_1 = \cdots = y_n =: y$. It follows that

$$x^{n-1} + (2a - x)^{n-1} - 2b = 0. \tag{4.9}$$

The coefficient of highest degree of this polynomial equation is 2 if n is odd, and $(n-1)2a$ if n is even. Hence (4.9) is not the zero polynomial. This shows that (4.9) has only finitely many solutions, which depend on a, b, m only.

Case 2: If not all of the numbers $x_1, \dots, x_n$ are equal, and hence $z_1 \neq z_2$, we put

$$l := |\{\iota \in \{1, \dots, n\} : x_\iota = z_1\}|.$$

Then $1 \leq l \leq n-1$ and $n - l = |\{\iota \in \{1, \dots, n\} : x_\iota = z_2\}|$. Then (4.8) yields that

$$z_1^{l-1} z_2^{n-l} + (2a - z_1)^{l-1}(2a - z_2)^{n-l} = 2b, \tag{4.10}$$

$$z_1^{l} z_2^{n-l-1} + (2a - z_1)^{l}(2a - z_2)^{n-l-1} = 2b. \tag{4.11}$$

If $l = 1$, then (4.10) gives

$$z_2^{n-1} + (2a - z_2)^{n-1} = 2b. \tag{4.12}$$

Since this is not the zero polynomial, there exist only finitely many possible solutions z_2. Furthermore, (4.11) gives

$$z_1[z_2^{n-2} - (2a - z_2)^{n-2}] = 2b - 2a(2a - z_2)^{n-2}.$$

If $z_2 \neq a$, then z_1 is determined by this equation. The case $z_2 = a$ cannot occur, since (4.12) with $z_2 = a$ implies that $a^{n-1} = b$, which is excluded by assumption.

If $l = n - 1$, we can argue similarly.

So let $2 \leq l \leq n-2$. Note that $0 < z_1, z_2 < 2a$ since $x_\iota, y_\iota > 0$ and $x_\iota + y_\iota = 2a$. Equating (4.10) and (4.11), we obtain

$$\left(\frac{2a - z_1}{z_1}\right)^{l-1} = \left(\frac{z_2}{2a - z_2}\right)^{n-l-1}. \tag{4.13}$$

The positive points on the curve $Z_1^{l-1} = Z_2^{n-l-1}$, where $Z_1, Z_2 > 0$, are parameterized by $Z_1 = t^{n-l-1}$ and $Z_2 = t^{l-1}$, $t > 0$. Therefore setting

$$t^{n-l-1} = (2a - z_1)/z_1, \quad t^{l-1} = z_2/(2a - z_2),$$

that is

$$z_1 = 2a/(1 + t^{n-l-1}), \quad z_2 = 2at^{l-1}/(1 + t^{l-1}), \tag{4.14}$$

we obtain a parameterization of the solutions z_1, z_2 of (4.13). Now we substitute (4.14) in (4.10) and thus get

$$(2a)^{n-1} \frac{t^{(l-1)(n-l)}}{(1 + t^{n-l-1})^{l-1}(1 + t^{l-1})^{n-l}} + (2a)^{n-1} \frac{t^{(l-1)(n-l-1)}}{(1 + t^{n-l-1})^{l-1}(1 + t^{l-1})^{n-l}} = 2b.$$

Multiplication by $(1+t^{n-l-1})^{l-1}(1+t^{l-1})^{n-l}$ yields a polynomial equation where the monomial of largest degree is

$$2bt^{(n-l-1)(l-1)}t^{(l-1)(n-l)},$$

and therefore the equation is of degree $(l-1)(2(n-l)-1)$. This equation will have at most $(l-1)(2(n-l)-1)$ positive solutions. Plugging these values of t into (4.14) gives a finite set of possible solutions of (4.10) and (4.11), depending only on a, b, m. This clearly results in a finite set of solutions of (4.8).

We turn to the case $2 \leq k \leq n-2$. We still assume that $x_1, \ldots, x_n$ are positive. By assumption and using Lemma 4.1, we get

$$(1+c^k)y_I = 2a \quad \text{and} \quad (1+c^{n-1})y_J = 2b$$

for $I, J \subset \{1, \ldots, n\}$, $|I| = k$, $|J| = n-1$, where $c > 0$ is a constant such that $x_\iota / y_\iota = c$ for $\iota = 1, \ldots, n$. We conclude that

$$y_{\tilde{I}} = \frac{b}{a}\frac{1+c^k}{1+c^{n-1}}$$

whenever $\tilde{I} \subset \{1, \ldots, n\}$, $|\tilde{I}| = n-1-k$. Since $1 \leq n-1-k \leq n-2$, we obtain $y_1 = \cdots = y_n =: y$. But then also $x_1 = \cdots = x_n =: x$. Thus we arrive at

$$x^k + y^k = 2a \quad \text{and} \quad x^{n-1} + y^{n-1} = 2b. \tag{4.15}$$

The set of positive real numbers x, y satisfying (4.15) is finite. In fact, (4.15) implies that

$$(2a - x^k)^{n-1} = y^{k(n-1)} = (2b - x^{n-1})^k,$$

and thus

$$\sum_{\iota=0}^{n-1} \binom{n-1}{\iota} (2a)^\iota (-1)^{n-1-\iota} x^{k(n-1-\iota)} - \sum_{\ell=0}^{k} \binom{k}{\ell} (2b)^\ell (-1)^{k-\ell} x^{(n-1)(k-\ell)} = 0. \tag{4.16}$$

The coefficient of the monomial of highest degree is $(-1)^{n-1} + (-1)^{k-1}$, if this number is nonzero, and otherwise it is equal to $(n-1)(2a)(-1)^{n-2}$, since $k(n-2) > (n-1)(k-1)$. In any case, the left side of (4.16) is not the zero polynomial, and therefore (4.16) has only a finite number of solutions, which merely depend on a, b, k, m.

Finally, we turn to the case where some of the numbers $x_1, \dots, x_n$ are zero. For instance, let $x_1 = 0$. Then we obtain that

$$y_1 y_{I'} = 2a, \quad y_1 y_{J'} = 2b$$

whenever $I', J' \subset \{2, \dots, n\}$, $|I'| = k-1$ and $|J'| = n-2$, and thus $y_{J'}/y_{I'} = b/a$. Therefore $y_{\tilde{I}} = b/a$ for all $\tilde{I} \subset \{2, \dots, n\}$ with $|\tilde{I}| = n-1-k$. Using that $k \geq 1$, we find that $y := y_2 = \dots = y_n = (b/a)^{1/(n-1-k)}$. Since $y_1 y^{k-1} = 2a$, we again get that $y_1, \dots, y_n$ can assume only finitely many values, depending only on $a, b, k, m = n-1$. ∎

4.2. Proof of Theorem 1.1 for $1 \leq i < j \leq n-2$. An application of Corollary 3.6 shows that, for $u \in \mathbb{S}^{n-1}$,

$$\wedge^i L_{h_0}(h)(u) + \wedge^i L_{h_0}(h)(-u) = 2\alpha \wedge^i \mathrm{id}_{u^\perp}, \tag{4.17}$$

$$\wedge^j L_{h_0}(h)(u) + \wedge^j L_{h_0}(h)(-u) = 2\beta \wedge^j \mathrm{id}_{u^\perp}, \tag{4.18}$$

Since $i < j \leq n-2$, Corollary 3.6 also implies that, for any fixed $u \in \mathbb{S}^{n-1}$, $L_{h_0}(h)(u)$ and $L_{h_0}(h)(-u)$ have a common orthonormal basis of eigenvectors.

Case 1: $\alpha^j \neq \beta^i$. We will show that there is a finite set, $\mathcal{F}^*_{\alpha,\beta,i,j}$, independent of u, such that

$$\det(L_{h_0}(h)(u)) = \frac{\det L(h)(u)}{\det L(h_0)(u)} \in \mathcal{F}^*_{\alpha,\beta,i,j}, \quad \text{for all } u \in \mathbb{S}^{n-1}. \tag{4.19}$$

Assume this is the case. Then, since h, h_0 are of class C^2, the function on the left-hand side of (4.19) is continuous on the connected set $\mathbb{S}^{n-1}$ and hence must be equal to a constant $\lambda \geq 0$. If $\lambda = 0$, then $\det L(h) \equiv 0$ and, as $\det L(h)$ is the density of the surface area measure $S_{n-1}(K, \cdot)$ with respect to spherical Lebesgue measure, this implies that the surface area measure $S_{n-1}(K, \cdot) \equiv 0$. But this cannot be true, since K is a convex body (with nonempty interior). Therefore $\lambda > 0$. Again using that $\det L(h)(u)$ is the density of the surface measure $S_{n-1}(K, \cdot)$, and similarly for h_0 and K_0, we obtain $S_{n-1}(K, \cdot) = S_{n-1}(\lambda^{1/(n-1)} K_0, \cdot)$. But then Minkowski's inequality and its equality condition imply that K and K_0 are homothetic (see [21, Thm 7.2.1]).

To construct the set $\mathcal{F}^*_{\alpha,\beta,i,j}$, we first put 0 in the set. Then we only have to consider the points $u \in \mathbb{S}^{n-1}$ where $\det L_{h_0}(h)(u) \neq 0$. At these points (4.17) and (4.18) show that the assumptions of Lemma 4.2 are satisfied (with n replaced by $n-1$). Hence there is a finite set $\mathcal{F}_{\alpha,\beta,i,j}$, such that for any $u \in \mathbb{S}^{n-1}$ with $\det L_{h_0}(h)(u) \neq 0$, if $x_1, \dots, x_{n-1}$ are the eigenvalues of $L_{h_0}(h)(-u)$ and

$y_1, \ldots, y_{n-1}$ are the eigenvalues of $L_{h_0}(h)(u)$, then $y_1, \ldots, y_{n-1} \in \mathcal{F}_{\alpha,\beta,i,j}$. Let $\mathcal{F}^*_{\alpha,\beta,i,j}$ be the union of $\{0\}$ with the set of all products of $n-1$ numbers each from the set $\mathcal{F}_{\alpha,\beta,i,j}$.

Case 2: If $\alpha^j = \beta^i$, then the assumptions can be rewritten in the form

$$\left(\frac{V_j(K_0|U)}{V_j(K|U)}\right)^{1/j} = \left(\frac{V_i(K_0|L)}{V_i(K|L)}\right)^{1/i} \tag{4.20}$$

for all $U \in \mathbb{G}(n,j)$ and all $L \in \mathbb{G}(n,i)$. Let $U \in \mathbb{G}(n,j)$ be fixed. By homogeneity we can replace K_0 by μK_0 on both sides of (4.20), where $\mu > 0$ is chosen such that $V_j(\mu K_0|U) = V_j(K|U)$. We put $M_0 := \mu K_0|U$ and $M := K|U$. Then, for any $L \in \mathbb{G}(n,i)$ with $L \subset U$, we have

$$V_j(M) = V_j(M_0) \quad \text{and} \quad V_i(M|L) = V_i(M_0|L).$$

By the theorem stated in the introduction of [5] (in [10, § 4] the authors review the results of [5] and give a somewhat shorter proof) this implies that M is a translate of M_0 and therefore $K|U$ and $K_0|U$ are homothetic. Since $j \geq 2$, Theorem 3.1.3 in [7] shows that K and K_0 are homothetic. ∎

5. The cases $2 \leq i < j \leq n-1$ with $i \neq n-2$

5.1. Existence of relative umbilics. We need another lemma concerning polynomial relations.

Lemma 5.1: *Let $n \geq 5$, $k \in \{2, \ldots, n-3\}$, $\gamma > 0$, and let positive real numbers $0 < x_1 \leq x_2 \leq \cdots \leq x_{n-1}$ be given. Assume that*

$$x_I + x_{I^*} = 2\gamma \tag{5.1}$$

for all $I \subset \{1, \ldots, n-1\}$, $|I| = k$, where $I^* := \{n-i : i \in I\}$. Then $x_1 = \cdots = x_{n-1}$.

Proof: Choosing $I = \{1, 2, \ldots, k\}$ in (5.1), we get

$$x_1 x_2 \cdots x_k + x_{n-k} \cdots x_{n-2} x_{n-1} = 2\gamma. \tag{5.2}$$

Choosing $I = \{1, n-k, \ldots, n-2\}$ in (5.1), we obtain

$$x_1 x_{n-k} \cdots x_{n-2} + x_2 \cdots x_k x_{n-1} = 2\gamma. \tag{5.3}$$

Subtracting (5.3) from (5.2), we arrive at

$$x_{n-k} \cdots x_{n-2}(x_{n-1} - x_1) + x_2 \cdots x_k (x_1 - x_{n-1}) = 0. \tag{5.4}$$

Assume that $x_1 \neq x_{n-1}$. Then (5.4) implies that

$$x_2 \cdots x_k = x_{n-k} \cdots x_{n-2}. \tag{5.5}$$

We assert that $x_2 = x_{n-2}$. To verify this, we first observe that $2 \leq k \leq n-3$ and $x_2 \leq \cdots \leq x_{n-2}$. After cancellation of factors with the same index on both sides of (5.5), we have

$$x_2 \cdots x_l = x_{n-l} \cdots x_{n-2}, \tag{5.6}$$

where $2 \leq l < n-l$ (here we use $k \leq n-3$). Since

$$x_l \leq x_{n-l}, \quad x_{l-1} \leq x_{n-l+1}, \quad \ldots \quad x_2 \leq x_{n-2},$$

equation (5.6) yields that $x_2 = \cdots = x_{n-2}$.

Now (5.2) turns into

$$x_1 x_2^{k-1} + x_2^{k-1} x_{n-1} = 2\gamma. \tag{5.7}$$

From (5.1) with $I = \{2, \ldots, k+1\}$ and using that $k \leq n-3$, we obtain

$$x_2^k + x_2^k = 2\gamma. \tag{5.8}$$

Hence (5.7) and (5.8) show that

$$x_1 + x_{n-1} = 2x_2. \tag{5.9}$$

Applying (5.1) with $I = \{1, \ldots, k-1, n-1\}$ and using (5.8), we get

$$2x_1 x_2^{k-2} x_{n-1} = 2\gamma = 2x_2^k,$$

hence

$$x_1 x_{n-1} = x_2^2. \tag{5.10}$$

But (5.9) and (5.10) give $x_1 = x_{n-1}$, a contradiction.

This shows that $x_1 = x_{n-1}$, which implies the assertion of the lemma. ∎

Proposition 5.2: *Let $K, K_0 \subset \mathbb{R}^n$ be convex bodies with K_0 centrally symmetric and of class C^2_+ and K having a C^2 support function. Let $n \geq 5$ and $k \in \{2, \ldots, n-3\}$. Assume that there is a constant $\beta > 0$ such that*

$$V_k(K|U) = \beta V_k(K_0|U)$$

for all $U \in \mathbb{G}(n,k)$. Then there exist $u_0 \in \mathbb{S}^{n-1}$ and $r_0 > 0$ such that

$$L_{h_0}(h)(u_0) = L_{h_0}(h)(-u_0) = r_0 \,\mathrm{id}_{T_{u_0}\mathbb{S}^{n-1}} .$$

Proof: For $u \in \mathbb{S}^{n-1}$, let $r_1(u), \ldots, r_{n-1}(u)$ denote the eigenvalues of the self-adjoint linear map $L_{h_0}(h)(u)\colon T_u\mathbb{S}^{n-1} \to T_u\mathbb{S}^{n-1}$, which are ordered such that

$$r_1(u) \le \cdots \le r_{n-1}(u).$$

Then we define a continuous map $R\colon \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ by

$$R(u) := (r_1(u), \ldots, r_{n-1}(u)).$$

By the Borsuk–Ulam theorem (cf. [13, p. 93] or [19]), there is some $u_0 \in \mathbb{S}^{n-1}$ such that

$$R(u_0) = R(-u_0). \tag{5.11}$$

Corollary 3.6 shows that $L_{h_0}(h)(u_0)$ and $L_{h_0}(h)(-u_0)$ have a common orthonormal basis $e_1, \ldots, e_{n-1} \in u_0^\perp$ of eigenvectors and by Lemma 3.4 at least one of $L_{h_0}(h)(u_0)$ or $L_{h_0}(h)(-u_0)$ is nonsingular. But $R(u_0) = R(-u_0)$ implies that $L_{h_0}(h)(u_0)$ and $L_{h_0}(h)(-u_0)$ have the same eigenvalues and thus they are both nonsingular. Therefore the eigenvalues of both $L_{h_0}(h)(u_0)$ and $L_{h_0}(h)(-u_0)$ are positive.

We can assume that, for $\iota = 1, \ldots, n-1$, e_ι is an eigenvector of $L_{h_0}(h)(u_0)$ corresponding to the eigenvalue $r_\iota := r_\iota(u_0)$. Next we show that e_ι is an eigenvector of $L_{h_0}(h)(-u_0)$ corresponding to the eigenvalue $r_{n-\iota}(-u_0)$. Let $\tilde{r}_\iota$ denote the eigenvalue of $L_{h_0}(h)(-u_0)$ corresponding to the eigenvector e_ι, $\iota = 1, \ldots, n-1$. Since $\tilde{r}_1, \ldots, \tilde{r}_{n-1}$ is a permutation of $r_1(-u_0), \ldots, r_{n-1}(-u_0)$, it is sufficient to show that $\tilde{r}_1 \ge \cdots \ge \tilde{r}_{n-1}$. By Corollary 3.6, for any $1 \le i_1 < \cdots < i_k \le n-1$ we have

$$(\wedge^k L_{h_0}(h)(u_0) + \wedge^k L_{h_0}(h)(-u_0)) e_{i_1} \wedge \cdots \wedge e_{i_k} = 2\beta e_{i_1} \wedge \cdots \wedge e_{i_k},$$

and therefore

$$r_{i_1} \cdots r_{i_k} + \tilde{r}_{i_1} \cdots \tilde{r}_{i_k} = 2\beta. \tag{5.12}$$

For $\iota \in \{1, \ldots, n-2\}$, we can choose a subset $I \subset \{1, \ldots, n-1\}$ with $|I| = k-1$ and $\iota, \iota+1 \notin I$, since $k+1 \le n-1$. Then (5.12) yields

$$r_I r_\iota + \tilde{r}_I \tilde{r}_\iota = r_I r_{\iota+1} + \tilde{r}_I \tilde{r}_{\iota+1} \ge r_I r_\iota + \tilde{r}_I \tilde{r}_{\iota+1},$$

which implies that $\tilde{r}_\iota \geq \tilde{r}_{\iota+1}$.

Let $1 \leq i_1 < \cdots < i_k \leq n-1$ and $I := \{i_1, \ldots, i_k\}$. Applying the linear map $\wedge^k L_{h_0}(h)(u_0) + \wedge^k L_{h_0}(h)(-u_0)$ to $e_{i_1} \wedge \cdots \wedge e_{i_k}$, we get

$$\prod_{\iota \in I} r_\iota(u_0) + \prod_{\iota \in I} r_{n-\iota}(-u_0) = 2\beta. \tag{5.13}$$

From (5.11) and (5.13) we conclude that the sequence

$$0 < r_1(u_0) \leq \cdots \leq r_{n-1}(u_0)$$

satisfies the hypothesis of Lemma 5.1. Thus, $r_1(u_0) = \cdots = r_{n-1}(u_0) =: r_0$. But $R(-u_0) = R(u_0)$ implies that also $r_1(-u_0) = \cdots = r_{n-1}(-u_0) = r_0$, which yields the assertion of the proposition. ∎

5.2. Proof of Theorem 1.1: remaining cases. It remains to consider the cases where $j = n-1$. Hence, we have $2 \leq i \leq n-3$. Proposition 5.2 implies that there is some $u_0 \in \mathbb{S}^{n-1}$ such that the eigenvalues of $L_{h_0}(h)(u_0)$ and $L_{h_0}(h)(-u_0)$ are all equal to $r_0 > 0$. But then Corollary 3.6 shows that

$$r_0^i + r_0^i = 2\alpha = 2\frac{V_i(K|L)}{V_i(K_0|L)},$$

for all $L \in \mathbb{G}(n,i)$, and

$$r_0^j + r_0^j = 2\beta = 2\frac{V_j(K|U)}{V_j(K_0|U)},$$

for all $U \in \mathbb{G}(n,j)$. Hence, we get

$$\Big(\frac{V_j(K_0|U)}{V_j(K|U)}\Big)^{1/j} = \Big(\frac{V_i(K_0|L)}{V_i(K|L)}\Big)^{1/i}$$

for all $U \in \mathbb{G}(n,j)$ and all $L \in \mathbb{G}(n,i)$. Thus again equation (4.20) is available and the proof can be completed as before. ∎

5.3. Proof of Corollary 1.3. Let K have constant width w. Then, [2, §64], the diameter of K is also w and any point $x \in \partial K$ is the end-point of a diameter of K. That is, there is $y \in \partial K$ such that $|x-y| = w$. Then K is contained in the closed ball $B(y,w)$ of radius w centered at y and $x \in \partial B(y,w) \cap K$. Thus if ∂K is C^2, then ∂K is internally tangent to the sphere $\partial B(y,w)$ at x. Therefore all the principle curvatures of ∂K at x are greater than or equal to the principle curvatures of $\partial B(y,w)$ at x, and thus all the principle curvatures of ∂K at x are at least $1/w$. Whence the Gauss–Kronecker curvature of ∂K at x is at least $1/w^{n-1}$. As x was an arbitrary point of ∂K this shows that if ∂K is a C^2 submanifold of $\mathbb{R}^n$ and K has constant width, then ∂K is of class C^2_+. Corollary 1.3 now follows directly from Corollary 1.2. ∎

6. Bodies of revolution

We now give a proof of Proposition 1.4. By assumption, there are constants $\alpha, \beta > 0$ such that

$$V_i(K|L) = \alpha V_i(K_0|L) \quad \text{and} \quad V_{n-1}(K|U) = \beta V_{n-1}(K_0|U),$$

for all $L \in \mathbb{G}(n,i)$ and $U \in \mathbb{G}(n,n-1)$, where $i \in \{1, n-2\}$. We can assume that the axis of revolution contains the origin and has direction $e \in \mathbb{S}^{n-1}$. Let $u \in \mathbb{S}^{n-1} \setminus \{\pm e\}$. Then there are $\varphi \in (-\pi/2, \pi/2)$ and $v_0 \in \mathbb{S}^{n-1} \cap u^\perp$ such that $u = \cos\varphi v_0 + \sin\varphi e$. For the sake of completeness we include a proof of the following lemma.

Lemma 6.1: *The map $L(h_K)(u)$ is a multiple of the identity map on $e^\perp \cap v_0^\perp$ and has $-\sin\varphi v_0 + \cos\varphi e$ as an eigenvector.*

Proof: By rotational invariance, there is some $r(\varphi) > 0$ such that

$$h_K(\cos\varphi v + \sin\varphi |v| e) = r(\varphi)|v|, \tag{6.1}$$

for all $v \in e^\perp$. Differentiating (6.1) twice with respect to $v \in e^\perp$ yields that, for any $v, w \in e^\perp \cap v_0^\perp$,

$$\cos^2\varphi d^2 h_K(\cos\varphi v_0 + \sin\varphi e)(v, w) = \tilde{r}(\varphi)\langle v, w\rangle,$$

where $\tilde{r}(\varphi) = r(\varphi) - \sin\varphi\, dh(\cos\varphi v_0 + \sin\varphi e)(e)$. Moreover, differentiating (6.1) with respect to v, we obtain, for any $v \in e^\perp \cap v_0^\perp$,

$$dh_K(\cos\varphi v_0 + \sin\varphi e)(v) = 0. \tag{6.2}$$

Differentiating (6.2) with respect to φ, we obtain

$$d^2 h_K(\cos\varphi v_0 + \sin\varphi e)(v, -\sin\varphi v_0 + \cos\varphi e) = 0.$$

Thus, if $v_1, \ldots, v_{n-2}$ is an orthonormal basis of $e^\perp \cap v_0^\perp$, then $-\sin\varphi v_1 + \cos\varphi e$, $v_1, \ldots, v_{n-2}$ is an orthonormal basis of eigenvectors of $L(h_K)(u)$ with corresponding eigenvalues x_1 and $x_2 = \cdots = x_{n-1} =: x$. ∎

Proof of Proposition 1.4: Let K and K_0 be as in Proposition 1.4 and let e be a unit vector in the direction of the common axis of rotation of K and K_0. Let h be the support function of K and h_0 the support function of K_0. Let $u \in \mathbb{S}^{n-1} \cap e^{\perp}$ be a point in the equator of $\mathbb{S}^{n-1}$ defined by e. As e is orthogonal to u, the vector e is in the tangent space to $\mathbb{S}^{n-1}$ at u. Let $e_2, \dots, e_{n-1}$ be an orthonormal basis for $\{u, e\}^{\perp}$. Then $e, e_2, \dots, e_{n-1}$ is an orthonormal basis for both $T_u\mathbb{S}^{n-1}$ and $T_{-u}\mathbb{S}^{n-1}$. By Lemma 6.1 there are eigenvalues x_1, and $x_2 = x_3 = \cdots = x_{n-1} =: x$ such that $L(h)(u)e = x_1 e$ and $L(h)(u)e_j = xe_j$ for $j = 2, \dots, n-1$. By rotational symmetry we also have $L(h)(-u)e = x_1 e$ and $L(h)(-u)e_j = xe_j$ for $j = 2, \dots, n-1$. Likewise, if y_1, and $y_2 = y_3 = \cdots = y_{n-1} =: y$ are the eigenvalues of $L(h_0)(u)$, then they are also the eigenvalues of $L(h_0)(-u)$ and $L(h_0)(\pm u)e = y_1 e$ and $L(h_0)(\pm u)e_j = ye_j$ for $j = 2, \dots, n-1$. By Proposition 3.5 the polynomial relations

$$\begin{aligned} x_1 x^{i-1} + x_1 x^{i-1} &= 2\alpha y_1 y^{i-1}, \\ x^i + x^i &= 2\alpha y^i, \\ x_1 x^{n-2} + x_1 x^{n-2} &= 2\beta y_1 y^{n-2} \end{aligned}$$

hold. The first two of these yields that $x/y = x_1/y_1$ and therefore

$$\alpha^{n-1} = (x/y)^{i(n-1)} = \beta^i.$$

As in the proof of Case 2 of the proof of Theorem 1.1 this gives that equation (4.20) holds which in turn implies that K and K_0 are homothetic. ∎

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